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Global Optimization Conditions for Certain Nonconvex Minimization Problems

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Abstract. By means of suitable dual problems to the following global optimization problems: extremum $\{f(x) : x \in M \subset X\}$, where f is a proper convex and lower-semicontinuous function and M a nonempty, arbitrary subset of a reflexive Banach space X, we derive necessary and sufficient optimality conditions for a global minimizer. The method is also applicable to other nonconvex problems and leads to at least necessary global optimality conditions.

Key words: Global optimization, necessary and sufficient optimality conditions.

Mathematics Subject Classifications (1991). 49K27, 90C26, 90C48.

1. Introduction

A great number of practically important issues in sciences, economics and engineering leads to constrained global optimization problems. In recent years a rapidly growing number of proposals has been published for the numerical solving of specific classes of multiextremal global optimization problems, for example Ritter [22], Hoffmann [13], Benacer and Tao [2], Snyman and Fatti [23], Tao [26], Pardalos and Rosen [19], Ratschek and Rokne [21], Horst *et al.* [16], Horst and Tuy [14], Bulatov [3], Mikhalevich *et al.* [18] and Chichinadze [30]. Large classes of global optimization were examined in the last years, among others the Lipschitz programming and the d.c.-programming. See for instance Horst [15], Horst and Tuy [14], Hiriart-Urruty [11], Tuy [28], Pinter [20], Dem'yanov and Vasil'ev [7], Toland [29], Auchmuty [1], Tao [26] and Craven *et al.* [6].

Development of further numerical methods for global optimization problems is associated with the search for necessary and sufficient optimality conditions. There exist several criteria for a global minimizer of a nonconvex function, see Horst and Tuy [14]. These criteria possess also a global character.

A method for solving global optimization problems is based on the Ψ-transformation of Chichinadze [30] and is related to the so-called integral methods, see Galperin and Zheng [9] and Chew and Zheng [4]. Hiriart-Urruty [11] gives necessary and sufficient optimality conditions for a global minimum of a d.c. function in terms of the ε-subdifferentials of both convex functions, compare also Hiriart-Urruty and Lemarechal [12]. By means of the results of convex analysis, as Hiriart-Urruty and Lemarechal, also Strekalovskiy [24] investigates nonconvex optimization problems and proves necessary and sufficient optimality conditions by means of a consideration of the normal cones of certain sets. He obtains new global optimality criteria for problems of the form

$$\text{extremum}\,\{f(x) : x \in M\}\,, \tag{1.1}$$

where f is a proper convex and closed function and M a nonempty arbitrary subset of a reflexive Banach space.

Inspired through Strekalovskiy [24], in this paper we also investigate the problems (1.1) and prove necessary and sufficient optimility conditions for a global minimizer or maximizer of (1.1). In contrast to Strekalovskiy [24] we make use of suitable dual problems of (1.1). We use the idea of the representation of a convex function as a supremum of a family of affin-linear functions, see Dem'yanov and Vasil'ev [7], and construct a dual problem. Then one can prove that a duality gap cannot occur. This method of using a dual problem immediately leads to sharp necessary conditions for global optimality and is also applicable to problems (1.1) with nonconvex functions, where the duality relation must be fulfilled. The sufficiency of the corresponding optimality conditions one can prove by means of changed mean valued theorems for convex functions.

In this way we obtain in the case of the problem

$$\sup\,\{f(x) : x \in M\} \tag{1.2}$$

the same optimality condition as Strekalovskiy [24], in the second case but another condition. This other condition is of a different kind as Strekalovskiy's corresponding condition and it is hard to compare these conditions because the methods of proofs are different. Applications and examples for the obtained results are given, where we notice that the application of Strekalovskiy's optimality condition for (1.2) on the problems of mathematical programming are given in Strekalovskiy [24].

2. Minimization of a Convex Function over an Arbitrary Nonempty Set

Let X be a real reflexive Banach space with the dual space X^*. The canonical bilinear form on $X \times X^*$ is denoted by $\langle .,. \rangle$.

Throughout, we shall use the following notations. Let M be a nonempty subset of X, then bd M, cl M and int M means the boundary, closure and interior of M, respectively.

Let the function $f : X \to R \cup \{+\infty\}$ be a lower-semicontinuous proper convex function, we denote with

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - z \rangle\}$$

the subdifferential of the function f at a point $x \in \text{dom}\, f$, the effective domain of f.

Then we consider as the first problem (P_1) the following minimization problem

$$\inf(P_1) := \inf_{x \in M} f(x) > -\infty\,, \tag{2.1}$$

and want to construct a necessary and sufficient optimality condition for a global minimizer $x_0 \in M$ for (2.1). For that reason we define to the primal problem (P_1) a corresponding dual problem (D_1)

$$\inf(D_1) := \inf_{z \in P} \inf_{x \in M} [f(z) + \langle p(x), x - z\rangle]\,. \tag{2.2}$$

Here the set $M \subset P$ is a subset of the set P and $p(x) \in \partial f(x)$ an arbitrary subgradient of f at the point $x \in X$. Then we can prove that a duality relation for (P_1) and (D_1) is fulfilled. $D(\partial f)$ denote the effective domain of the subdifferential mapping $\partial f : X \to 2^{X^*}$.

The problem (D_1) one can view as a dual problem of (P_1) because of the valid inequality for the convex function f and the subgradient of f as a dual element.

LEMMA 2.1. *If $M \subset D(\partial f)$ holds, then the duality relation*

$$\inf(P_1) = \inf(D_1) \tag{2.3}$$

is valid.

Proof. For $x \in X$ with $p(x) \in \partial f(x)$ the inequality

$$f(z) \geq f(x) + \langle p(x), z - x\rangle \qquad \forall z \in X$$

holds, it follows that the inequality for $x = z \in M$

$$\inf(P_1) \geq \inf_{x \in M} \inf_{z \in P} [f(z) + \langle p(x), x - z\rangle] \geq \inf(P_1)$$

is fulfilled. Therefore we have proven the duality relation (2.3) for each set $P \supset M$. □

COROLLARY 2.2. *If one chooses the set P in the following manner*

$$P := \bigcup_{w \in M} \{z \in X : f(z) = f(w)\}\,, \tag{2.4}$$

then $P \supset M$ is valid, we obtain as the dual problem (D_1)

$$\inf(D_1) = \inf_{w \in M} [f(w) + \inf_{f(z)=f(w)} \inf_{x \in M} \langle p(x), x - z\rangle] \tag{2.5}$$

and the duality relation (2.3) is also fulfilled.

The proof follows immediately from the assumptions. Now we use the dual problem (2.5) and the equation (2.3) to prove global optimality conditions for (2.1).

THEOREM 2.3. *Suppose that the assumption $M \subset D(\partial f)$ holds. If the element $x_0 \in M$ is a solution of the primal problem (P_1), then the following condition with $p(x) \in \partial f(x)$*

$$\langle p(x), x - z\rangle \geq 0 \quad \forall x \in M \textit{ and } \forall z \in X : f(z) = f(x_0) \tag{2.6}$$

holds.

Proof. If $x_0 \in M$ is a solution of (P_1), then we obtain by (2.3) and (2.5) the equation

$$f(x_0) = \inf_{w\in M} \left[f(w) + \inf_{f(z)=f(w)} \inf_{x\in M} \langle p(x), x - z\rangle\right].$$

From this with $w = x_0 \in M$ it follows that (2.6) is valid. □

By Ekeland and Temam [8, II, §2] the equivalence

$$\inf_{x\in M} f(x) = f(x_0) \Leftrightarrow \langle f'(x), x - x_0\rangle \geq 0 \quad \forall x \in M$$

holds for a convex and continuously Gâteaux-differentiable function f and a convex set M of restrictions. The condition (2.6) appears as a generalization of this known variational inequality for an arbitrary set M of restrictions.

LEMMA 2.4. *If the set M is additionally a convex set, then the necessary condition (2.6) for a global minimizer $x_0 \in M$ of (P_1) is also sufficient.*

Proof. From the optimality condition with $p(y) \in \partial f(y)$

$$\langle p(y), y - z\rangle \geq 0 \quad \forall y \in M \text{ and } \forall z \in X : f(z) = f(x_0)$$

it follows that with $z = x_0$ and $y = \lambda x + (1-\lambda)x_0$ for $x \in M$ and $\lambda \in [0,1]$ the inequality

$$\Big\langle p\big(x_0 + \lambda(x - x_0)\big), x - x_0\Big\rangle \geq 0 \quad \forall x \in M$$

and $p\big(x_0 + \lambda(x - x_0)\big) \in \partial f\big(x_0 + \lambda(x - x_0)\big)$ is valid. For $\lambda > 0$ and sufficiently small that the subdifferential $\partial g\big(x_0 + \lambda(x - x_0)\big)$ is especially a bounded set in the reflexive Banach space X^*. Therefore there exists a weakly convergent subsequence $p\big(x_0 + \lambda'(x - x_0)\big) \to p(x_0)$ with $x_0 + \lambda'(x - x_0) \to x_0$ as $\lambda' \to +0$ and because of the upper semicontinuity of the point-to-set mapping ∂f we obtain the inequality $\langle p(x_0), x - x_0\rangle \geq 0$ $\forall x \in M$. Due to $f(x) \geq f(x_0) + \langle p(x_0), x - x_0\rangle \geq f(x_0)$ $\forall x \in M$ holds, this implies the optimality of $x_0 \in M$ for (P_1). □

The condition (2.6) as a necessary and sufficient condition for a global minimizer $x_0 \in M$ also in the case of a convex set M of restrictions is new in the literature. Usually one can find the condition $\partial f(x_0) \cap \Gamma^+(x_0) \neq \emptyset$, see Dem'yanov and Vasil'ev [7], Sukharev *et al.* [25], where $\Gamma^+(x_0)$ is the conjugate cone to the cone of feasible directions of M at $x_0 \in X$.

THEOREM 2.3. *Suppose that the assumptions*

$$-\infty \leq \inf(f, X) < f(v) \qquad \forall v \in M \tag{2.7}$$

and

$$\{x \in X : f(x) \leq f(v)\} \subset \operatorname{int} \operatorname{dom} f \qquad \forall v \in M \tag{2.8}$$

are fulfilled. Then by the condition (2.6) it follows that $x_0 \in M$ is a global minimizer of (P_1).

Proof. We denote with $L_a := \{x \in X : f(x) \leq a\}$, $a \in R$, the level sets of the convex function f and introduce the function $\Phi : X \to R \cup \{+\infty\}$ given by

$$\Phi(x) = \inf_{f(z)=f(x)} \langle p(x), x - z \rangle .$$

By (2.3), (2.5) and the convexity of f it follows that the inequality

$$\Phi(x) \geq f(x) - f(x_0) \qquad \forall x \in X \tag{2.9}$$

holds.

Consider in the proof two cases for the convex and closed level set $L_c = \{x \in X : f(x) \leq f(x_0) = c\}$, which has a nonempty interior.

In the first case we suppose that L_c is an unbounded set. If we now assume that there exists an element $\xi \in M$ so that $f(\xi) < f(x_0) = f(x_0) = c$ holds, then we consider the continuous and convex real function $\varphi : [0, +\infty) \to R$ defined by

$$\varphi(t) = f\Big(z + t(\xi - z)\Big) - f(z) - t\,\langle p(\xi), \xi - z \rangle ,$$

with $p(\xi) \in \partial f(\xi)$ and $z \in X : f(z) = f(x_0) = c$. Then $\varphi(0) = 0$, $q(t) := \Big\langle p\Big(z+t(\xi-z)\Big)-p(\xi), \xi-z\Big\rangle \in \partial\varphi(t)$ for $t \geq 0, 0 \in \partial\varphi(1)$ and $q(t) \geq 0\ \forall t > 1$.

Furthermore $\varphi(1) = f(\xi) - f(z) - \langle p(\xi), \xi - z \rangle \leq 0$ is true. If $\varphi(1) = 0$ holds, we get $\langle p(\xi), \xi - z \rangle < 0$ and $\Phi(\xi) < 0$ in contrast to (2.6). If $\varphi(1) < 0$ is valid, then there exists a point $s > 1$ so that $\varphi(s) = 0$ is true. It follows that

$$f\Big(z + s(\xi - z)\Big) - f(z) = s\,\langle p(\xi), \xi - z \rangle \tag{2.10}$$

holds. To $\xi \in \operatorname{int} L_c$ one can find an element $z \in \operatorname{bd} L_c$ so that $z + t(\xi - z) \in L_c$ for all $t \geq 0$ is fulfilled due to the unboundness of the level set L_c. Therefore $z + s(\xi - z) \in \operatorname{int} L_c$, $f\Big(z + s(\xi - z)\Big) < f(z) = f(x_0) = c$ with $s > 1$ and by (2.10) $\varphi(\xi) < 0$ in contrast to (2.6).

In the second case we suppose that L_c is a bounded set. We now assume that also there exists an element $\xi \in M$ so that $f(\xi) < f(x_0) = c$ is true. Then there exists an element $\bar{x} \in X$ with

$$-\infty < \inf(f, X) = f(\bar{x}) < f(\xi) .$$

Consider the line $x(t) = t\,\bar{x} + (1-t)\xi$ in $X, t \in R$. The points $z_1 = t_1\,\bar{x} + (1-t_1)\xi$ with $t_1 < 0$ and $z_2 = t_x\,\bar{x} + (1 - t_2)\xi$ with $t_2 > 1$ are the intersection points of the line above and the boundary $\operatorname{bd} L_c$ of the bounded convex and closed level set L_c, so that $f(z_1) = f(z_2) = c$ is true.

With the convex and continuous function $\psi : [t_1, t_2] \to R$

$$\psi(t) = f(t\bar{x} + (1-t)\xi) - f(z_1) - (t - t_1)\,\langle p(\xi), \bar{x} - \xi\rangle$$

we obtain $\psi(0) = f(\xi) - f(z_1) + \langle p(\xi), z_1 - \xi\rangle \leq 0$. If $\psi(0) = 0$ holds, then $f(\xi) - f(z_1) = \langle p(\xi), \xi - z_1\rangle < 0$, contrary to (2.6).

If $\psi(0) < 0$ holds, then we have

$$q(t) := \Big\langle p\Big(\xi + t(\bar{x} - \xi)\Big) - p(\xi), \bar{x} - \xi\Big\rangle \in \partial\psi(t)\,, \qquad 0 \in \partial\psi(0) \text{ and}$$

$$\psi(t_2) = -(t_2 - t_1)\,\langle p(\xi), \bar{x} - \xi\rangle\,.$$

Now we consider the level set $L_\gamma = \{x \in X : f(x) \leq f(\xi) = \gamma\}$. By Ioffe and Tichomirov [16, 4.3, Hilfssatz 2] we obtain for the normal cone of the level set L_γ at the point $\xi \in \text{bd}\, L_\gamma$

$$N_{L_\gamma}(\xi) = \{\xi^* \in X^* : \langle \xi^*, x - \xi\rangle \leq 0 \;\; \forall x \in L_\gamma\} = \text{cone}\,\partial f(\xi)\,.$$

It follows that $\langle p(\xi), \bar{x} - \xi\rangle \leq 0$ and therefore $\psi(t_2) \geq 0$ is valid. If $\psi(t_2) = 0$ is true, then we have

$$f(\bar{x}) \geq f(\xi) + \langle p(\xi), \bar{x} - \xi\rangle = f(\xi)$$

in contrast to the assumption. For that reason $\psi(t_2) > 0$ is true and there exists an intermediate value

$$\psi(s) = f\Big(\xi + s(\bar{x} - \xi)\Big) - f(z_1) - (s - t_1)\,\langle p(\xi), \bar{x} - \xi\rangle = 0$$

with $s < t_2$. Due to $y := \xi + s(\bar{x} - \xi) \in \text{int}\, L_c$ it follows that

$$f(y) - f(z_1) = \langle p(\xi), \xi - y\rangle < 0$$

contrary to (2.6). □

EXAMPLE 2.6. Consider the function f with

$$f(x) = (x_1 + 1)^2 + (x_2 + 1)^2\,, \qquad x = (x_1, x_2)^T \in M = M_1 \cup M_2 \subset R^2\,,$$

where

$$M_1 = \{x \in R^2 : 0 \leq x_i \leq 1,\; i = 1, 2\}$$

$$M_2 = \{x \in R^2 : 2 \leq x_1 \leq 3,\; 0 \leq x_2 \leq 1\}\,.$$

For the problem $\inf_{x\in M} f(x)$ the points $v = (2,0)^T$ and $w = (0,0)^T$ are critical points. We check by means of condition (2.6) whether one of the points is a global minimizer of f over M.

For $v = (2,0)^T = z$ and $x = (x_1, x_2)^T = (1,0)^T \in M_1 \subset M$ we get $\langle f'(x), x - z\rangle = -4 < 0$. Therefore v is not a global minimizer of the problem above.

For $w = (0,0)^T$ and $f(w) = 2 = f(z) = (z_1 + 1)^2 + (z_2 + 1)^2$ we obtain with $z_1 = \sqrt{2}\cos\alpha - 1$ and $z_2 = \sqrt{2}\sin\alpha - 1$

$$\langle f'(x), x - z\rangle \geq 4\,[1 - \cos(\pi/4 - \alpha)] \geq 0 \qquad \forall \alpha \in R\,.$$

For that reason $w = (0,0)^T$ is a global minimizer of the problem.

In the sequel we consider the problem of mathematical programming with a convex cost function $f : X \to R$ and restrictions, which are locally Lipschitz continuous functions.

Let A be a nonempty closed subset of the reflexive Banach space X and let M be the set of following restrictions

$$M = \{x \in A \subset X : g_i(x) \leq 0,\ h_j(x) = 0,\ \forall_1^n i,\ \forall_1^m j\}$$

with the locally Lipschitz functions $g_i, h_j : X \to R$. Then we consider the problem

$$\inf_{x\in M} f(x) \tag{2.11}$$

of mathematical programming.

THEOREM 2.7. *Suppose that* $-\infty \leq \inf(f, X) < f(x_0)$ *for* $x_0 \in M$ *holds. The element* $x_0 \in M$ *is a global minimizer of (2.11) if and only if the condition*

$$\langle p(x), x - z\rangle \geq 0 \quad \forall x \in M\,,\ p(x) \in \partial f(x)\,,\ \forall z \in X : f(z) = f(x_0) \tag{2.12}$$

is fulfilled. If the condition (2.12) holds, then there exist real numbers $r_0, r_i \geq 0$ *and* $s_j \in R$, *not all zero, and a point* $x^* \in X^*$ *so that*

$$\left.\begin{aligned} &\text{(a)}\ \ r_i\, g_i(x_0) = 0 \quad i = 1, ..., n \\ &\text{(b)}\ \ x^* \in r_0\, \partial f(x_0) + \sum_1^n r_i\, \partial g_i(x_0) + \sum_1^m s_j\, \partial h_j(x_0) \\ &\text{(c)}\ \ -x^* \in \text{cone}\ \partial d_A(x_0) \end{aligned}\right\} \tag{2.13}$$

is true. Here the subdifferentials are understanding in the sense of Clarke [5] and d_A *is the distance function of the closed set A.*

Proof. By Theorem 2.5 we get the first part of the theorem with (2.12). If now (2.12) is valid, then the assertion (2.13) we obtain by the application of the multiplier rule of Clarke [5] to the problem $\inf\{\langle p(x), x - x_0\rangle : x \in M\}$. Here $\partial f(x_0) = \partial\varphi(x_0)$ is true for the function $\varphi : X \to R$ defined by $\varphi(x) = \langle p(x), x - x_0\rangle$, since the subdifferential mapping ∂f is a maximal monotone operator. □

EXAMPLE 2.8. Consider the function f with $f(x) = |x_2|$, $x = (x_1, x_2)^T \in M \subset R^2$, $M = \{x \in R^2 : g(x) \leq 0,\ h(x) = 0\}$, $h(x) = x_1^2 + x_2^2 - 4$ and

$$g(x) = \begin{cases} -x_1 + x_2 + 1 & \text{for } x_1 \geq 0 \\ x_1 + x_2 + 1 & \text{for } x_1 < 0 \end{cases}.$$

The points $u = (0,-2)^T$, $v = (2,0)^T$ and $w = (-2,0)^T$ satisfying (2.13) are classically critical points. The point u is not a global minimizer of f over M, since by (2.12) $\langle p(x), x - z\rangle = -2 < 0$ for $z = (z_1, 2)^T$ with $z_1 \in R$ and $x = (2,0)^T \in M$ is fulfilled. On the other hand the points v and w satisfy (2.12) and are global minimizers of f over M.

3. Minimization of a Concave Function over an Arbitrary Nonempty Set

In this chapter we examine the following optimization problem

$$\inf(P_2) := \inf_{x\in M}\,[-f(x)] > -\infty \tag{3.1}$$

and want to derive a necessary and sufficient optimality condition for a global minimizer of (P_2). Here, let $f : X \to R \cup \{+\infty\}$ be a proper convex and closed function on the reflexive Banach space X, $M \subset X$ a nonempty subset of X.

First we define again a dual problem (D_2) to (P_2) in a similar manner as in Section 2:

$$\inf(D_2) := \inf_{z\in P}\inf_{x\in M}\,[-f(x) + \langle p(z), z - x\rangle]\,. \tag{3.2}$$

Here let $P \supset M$ be an arbitrary subset of X and $p(z) \in \partial f(z)$ an arbitrary subgradient of the function f at the point $z \in \operatorname{dom} f$.

LEMMA 3.1. *If $P \subset D(\partial f)$ and $P \supset M$ hold, then the duality relation*

$$\inf(P_2) = \inf(D_2) \tag{3.3}$$

is true.

Proof. For $z \in X$ with $p(z) \in \partial f(z)$ we get

$$-f(z) + \langle p(z), z - x\rangle \geq -f(x) \qquad \forall x \in X$$

and $\inf(D_2) \geq \inf(P_2)$. Due to $P \supset M$ we obtain the equation (3.3). □

COROLLARY 3.2. *If we choose additional*

$$P := \bigcup_{w\in M} \{z \in X : f(z) = f(x)\}\,, \tag{3.4}$$

then we get as a dual problem (3.2)

$$\inf(D_2) = \inf_{w\in M}\,[-f(w) + \inf_{f(z)=f(w)}\,\inf_{x\in M}\,\langle p(z), z - x\rangle] \tag{3.5}$$

and the duality relation (3.3) is also fulfilled.

Proof. Because of $P \supset M$ and Lemma 3.1 the assertion is true. □

THEOREM 3.3. *Suppose that $P \subset D(\partial f)$ holds. If $x_0 \in M$ is a solution of (P_2), then the following condition*

$$\langle p(z), z - x\rangle \geq 0 \qquad \forall x \in M\,,\ \ \forall z \in X : f(z) = f(x_0) \tag{3.6}$$

is valid.

Proof. By (3.3) and (3.5) we have

$$-f(x_0) = \inf_{w \in M} \Big[-f(w) + \inf_{f(z)=f(w)} \ \inf_{x \in M} \langle p(z), z - x\rangle \Big]$$

$$-f(x_0) \leq -f(x_0) + \inf_{f(z)=f(x_0)} \ \inf_{x \in M} \langle p(z), z - x\rangle\,,$$

therefore (3.6) is fulfilled. □

THEOREM 3.4. *Suppose that the following assumptions*

$$-\infty \leq \inf(f, X) < f(v) \qquad \forall v \in M \tag{3.7}$$

and

$$\{x \in X : f(x) \leq f(v)\} \subset \operatorname{int}\operatorname{dom} f \qquad \forall v \in M \tag{3.8}$$

hold. If an element $x_0 \in M$ satisfy (3.6), then $x_0 \in M$ is a global minimizer of the problem (P_2).

Proof. If (3.6) is fulfilled for an element $x_0 \in M$ and there exists an element $\xi \in M$ with $-f(\xi) < -f(x_0) = -f(z)$, $z \in X$, then we consider the convex and continuous real function $\varphi : [0, +\infty) \to R$

$$\varphi(t) = f(t\,z + (1-t)\xi) - f(\xi) - t\,\langle p(z), z - \xi\rangle$$

with $z \in X : f(z) = f(x_0)$ and $p(z) \in \partial f(z)$. For this function $\varphi(0) = 0$, $q(t) := \Big\langle p\big(\xi + t(z-\xi)\big) - p(z), z - \xi\Big\rangle \in \partial\varphi(t), 0 \in \partial\varphi(1)$, $q(t) \geq 0 \ \forall t \geq 1$ and $\varphi(1) = f(z) + \langle p(z), \xi - z\rangle - f(\xi) \leq 0$ hold. If $\varphi(1) = 0$ is valid, then $f(z) - f(\xi) = \langle p(z), z - \xi\rangle < 0$ holds contrary to (3.6). Therefore $\varphi(1) < 0$ is true. We denote by $L_c = \{x \in X : f(x) \leq f(x_0) = f(z) = c\}$ the level set of the function f. The level set L_c is convex and closed and possesses by the assumptions above a nonempty interior. There exists an unique determined element $\zeta \in \operatorname{bd} L_c$ of the minimal distance of the element $\xi \in M$ to L_c : $\inf\{\|z-\xi\| : z \in L_c\} = \|\zeta - \xi\|$. We consider in the sequel the function φ with $z = \zeta$. If $\varphi(t) < 0$ for all $t \geq 1$ holds, then $\varphi(t) = \varphi(1) < 0 \ \ \forall t \geq 1$ is valid and we obtain for sufficient small $t > 1$ with $\xi + t(\zeta - \xi) \in \operatorname{int} L_c$ the inequality

$$f\big(\xi + t(\zeta - \xi)\big) - f(\zeta) = (t-1)\,\langle p(\zeta), \zeta - \xi\rangle < 0$$

contrary to (3.6). Therefore there must exist a number $s > 1$, so that $\Phi(s) = 0$ and consequently

$$f\big(\xi + s(\zeta - \xi)\big) - f(\xi) = s\,\langle p(\zeta), \zeta - \xi\rangle$$

is valid. If L_c is unbounded along the half line $\{\xi + t(\zeta - \xi) : t \geq 0\}$, then $\xi + s(\zeta - \xi) \in \operatorname{int} L_c$ and $\langle p(\zeta, \zeta - \xi\rangle < 0$ contrary to (3.6). On the other hand in the case that L_c is a bounded set along the half line above there exists a boundary point $y = r\zeta + (1-r)\xi \in \operatorname{bd} L_c$ with $r > 1$ of the level set L_c. We calculate $\varphi(r) = f(y) - f(\xi) - r\,\langle p(\zeta), \zeta - \xi\rangle$ and if $\langle p(\zeta, \zeta - \xi\rangle \geq 0$ holds, then $\varphi(r) \leq \varphi(1)$ contrary to $0 \in \partial\varphi(1)$ and to the definition of the element $\zeta \in \operatorname{bd} L_c$. Consequently $\langle p(\zeta), \zeta - \xi\rangle < 0$ holds in contradiction to (3.6) and the assertion of the theorem is true. □

4. Concluding remarks

The method to use a suitable dual problem for the construction of necessary and sufficient optimality conditions for a global minimum is also applicable to problems with nonconvex cost functions. One may obtain, at least, sharp necessary optimality conditions for a global minimum. This method involves a constructive possiblity for optimality conditions. We now shall apply this method to obtain optimality conditions of d.c. functions for selected examples.

Let $F : X \to R$ be a d.c. function, $F = f - g$ with the convex and continuous functions $f, g : X \to R$. For $p(y) \in \partial f(y)$ and $q(x) \in \partial g(x)$ we get

$$F(x) + \langle p(y) - q(x), y - x\rangle \geq F(y) \qquad \forall x, y \in X\,. \tag{4.1}$$

If $x_0 \in X$ is a global minimizer of the problem $\inf(F, X) = F(x_0)$, then from (4.1) it follows that the condition

$$\langle p(y) - q(x), y - x\rangle \geq 0 \qquad \forall y \in X\,,\ \ \forall x \in X : F(x) = F(x_0) \tag{4.2}$$

is fulfilled.

To prove (4.2), compare the proofs of Lemma 2.1, Corollary 2.2 and Theorem 2.3. In connection with the optimality condition (4.2) we also consider the function $\Phi : X \to R$

$$\Phi(y) = \inf_{F(x)=F(x_0)} \langle p(y) - q(x), y - x\rangle$$

and obtain by (4.1) the inequality

$$\Phi(y) \geq F(y) - F(x_0) \qquad \forall y \in X\,. \tag{4.3}$$

For a d.c. function (4.2) is in general merely a necessary optimality condition for a global minimizer.

The function $F : R \to R$, $F(x) = 1/(x^2+1) = f(x) - g(x)$ with the convex functions $f(x) = x^2 + 1/(x^2+1)$ and $g(x) = x^2$ the condition (4.2) is fulfilled for the point $x_0 = 0$, the absolute maximum point of F : $[f'(y) - g'(x_0)]\,(y - x_0) \geq 0$ $\forall y \in R$.

The function $F : R \to R$, $F(x) = (x^2-1)^3 = f(x) - g(x)$ with the convex functions $f(x) = (x^2-1)^3 + 2x^4$ and $g(x) = 2x^4$ fulfil for the points $x = +1$ and $x = -1$, two saddle points of F, also the condition (4.2). For the function Φ with

$$\Phi(y) = \inf_{F(x)=0} [f'(y) - g'(x)] (y - x)] ,$$

$\Phi(y) \geq 0 \ \forall y \in R$ is valid, but $F(x) < 0 \ \forall x \in R : |x| < 1$ holds. The absolute minimum point of F is $x_0 = 0$ with $F(0) = -1$. On the other hand, for a local minimum point of a d.c. function (4.2) is generally not true. Let us consider the d.c. function $F : R \to R$, $F(x) = 0.001 [\cosh x - (x - 0.2)^4]$. We have $F(x_i) = 0$, $i = 1, 2, 3, 4$, with $x_1 = -0,8927$, $x_2 = 1,45151$, $x_3 = -9,9715$, $x_4 = 9,6971$, the absolute minimum point is $(x_5, F(x_5)) = (-8,6056\,;\ -3,2812)$, the local maximum point is $(x_6, F(x_6)) = (0,8079\,;\ 0,0012)$ and the local minimum point is $(x_7, F(x_7)) = (8,3871\,;\ -2,2978)$.

The condition (4.2) is for this function a necessary and sufficient optimality condition for the global minimizer. For the local extremums (4.2) is not valid.

We investigate the behaviour of the d.c. functions more precisely and extensively in another paper.

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